GROUP - A

1. Write the definite integral which is equal to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{\sqrt{n^2+r^2}}$$ **[2018(A)**

Ans. We know that $\int_a^b f(x)\,dx$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(a+\frac{r}{n}\right),\ \text{where } b = a + nh$$

Now $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{\sqrt{n^2+r^2}}$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)}{\sqrt{1+\left(\frac{r}{n}\right)^2}}$$

$$= \int_0^1 \frac{x\,dx}{\sqrt{1+x^2}}\ \text{(Let } 1+x^2=t^2,\ 2x\,dx = 2t\,dt.)$$

$$= \int_1^2 \frac{t\,dt}{t} = \int_1^2 dt = t\Big]_1^2 = 2-1 = 1$$

2. If p and q are respectively degree and order of the differential equation $y = e^{dy/dx}$, then write the relation between p and q. **[2018(A)**

Ans. Given differential equation is

$$y = e^{\frac{dy}{dx}} \Rightarrow \frac{dy}{dx} = \ln y$$

Whose order = 1 = p

Degree = 1 = q

$\therefore\ p = q$

3. Write the value of $\int_0^1 \{x\}\,dx$ where $\{x\}$ stands for fractional part of x. **[2017(A)**

Ans. $I = \int_0^1 \{x\}\,dx$

$$= \int_0^1 \{x-[x]\}\,dx$$

$$= \int_0^1 (x-0)\,dx = \frac{x^2}{2}\Big]_0^1 = \frac{1}{2}$$

4. Write the order of the differential equation of the family of circles :-

$ar^2 + ay^2 + 2gx + 2fy + c = 0$

$ax^2 + ay^2 + 2gx + 2fy + c = 0$

[2017(A)

Ans. As there are 3 independent constants, the order of the differential equation is 3.

5. Write the value of :

$$\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx - \int_0^{\pi/2}\frac{\cos x}{\sin x+\cos x}dx$$

[2016 (A)

Ans. $\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}dx - \int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}dx$

$$= \int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx - \int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}dx$$

$$= \int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}dx - \int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}dx = 0$$

6. If p and q are the order and degree of the differential equation

$$y\left(\frac{dy}{dx}\right)^2 + x^2\frac{d^2y}{dx^2} + xy = \sin x,$$

then choose the correct statement out of (i) p > q, (ii) p = q, (iii) p < q. **[2016 (A)**

Ans. Order of the given differential = p = 2

Degree of the given differential equation = q = 1

$\therefore\ p > q$

7. If $\int_2^3 f(z)dz = 9$, then write the value of

$$\int_2^3 f(\phi(z))d(\phi(z)).$$ **[2015 (A)**

Ans. $\int_2^3 f(z)dz = 9$

Let $I = \int_2^3 f(\phi(z))d(\phi(z))$

$= \int_2^3 f(z)dz = 9 \left(\because \int_a^b f(x)dx = \int_a^b f(t)dt \right)$

8. Write the order of the differential equation of the system of ellipses :

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **[2015 (A)**

Ans. As there are two unknown constants in the system of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the order of the differential equation is 2.

9. What do you mean by integration ? Write your answer in one sentence. [2014 (A)

Ans. Integration is the antiderivative of a function.

10. Write the differential equation of the family of straight lines parallel to the y-axis. [2014 (A)

Ans. $\frac{dx}{dy} = 0$ is the differential equation of family of lines parallel to y-axis.

11. Write the value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^5 x \cos x dx$. [2013 (A)

Ans. Let $f(x) = \sin^5 x \cos x$

$f(-x) = \sin^5(-x) \cos(-x)$

$= -\sin^5 x \cos x = -f(x)$

i.e. f is an odd function.

Thus $\int_{-\pi/4}^{\pi/4} \sin^5 x \cos x dx = 0$

12. Write the degree of the differential equation

$ln\left(\frac{d^2y}{dx^2}\right) = y$ **[2013 (A)**

Ans. The degree of the differential equation $ln\left(\frac{d^2y}{dx^2}\right) = y$ is 1

13. What is $F'(t)$ if $F(t) = \int_a^t e^{3x} \cdot \cos 2x\, dx$? [2012(A)

Ans. $F'(t) = \int_a^t e^{3x} \cos 2x dx \Rightarrow F^1(t) = e^{3t} \cos 2t.$

14. Integrate

$\int \frac{3 + \cos x + \tan^2 x}{2x + \sin x + \tan x}$ **[2012(A)**

Ans. Let $I = \int \frac{3 + \cos x + \tan^2 x}{2x + \sin x + \tan x} dx$

$= \int \frac{3 + \cos x + \sec^2 x - 1}{2x + \sin x + \tan x} dx$

$= \int \frac{2 + \cos x + \sec^2 x}{2x + \sin x + \tan x} dx$

Let $2x + \sin x + \tan x = t$

$\Rightarrow (2 + \cos x + \sec^2 x)\, dx = dt$

$\therefore I = \int \frac{dt}{t} = \ln|t| + C$

$= \ln|2x + \sin x + \tan x| + C$

15. Write the particular solution of the equation $\frac{dy}{dx} = \sin x$ given that $y(\pi) = 2$. [2012(A), 2011(A)

Ans. $\frac{dy}{dx} = \sin x \Rightarrow \int dy = \int \sin x\, dx$

$\Rightarrow y = -\cos x + C$

Using the given condition

$x = \pi \Rightarrow y = 2$ we get.

$2 = 1 + C \Rightarrow C = 1$

$\therefore$ The required solution is $y = -\cos x + 1$

16. Write the order and degree of the following differential equation : [2012(A)

$$\frac{d^2y}{dx^2} = \frac{2y^3 + \left(\frac{dy}{dx}\right)^4}{\sqrt{\frac{d^2y}{dx^2}}}$$

Ans. Order = 2, Degree = 3

17. $\int \frac{\cot x\, dx}{ln \sin x} = ?$ **[2011(A)**

Ans. $\int \frac{\cot x\, dx}{ln \sin x} = ln(ln \sin x) + C$

18. What is $F'(x)$ if $F(x) = \int_0^x e^{2t} \sin 3t\, dt$? [2011(A), 2009(A)

Ans. If $F(x) = \int_0^x e^{2t} \sin 3t\, dt$

then $F'(x) = e^{2x} \sin 3x$

19. $\int \frac{dx}{\cos^2 x \sin^2 x} = ?$ [2011(A)]

Ans. $\int \frac{dx}{\cos^2 x \sin^2 x} = 4\int \frac{dx}{\sin^2 2x}$

$= 4\int \operatorname{cosec}^2 2x\, dx = -2 \cot 2x + C$

20. What is the value of

$\int \frac{d}{dx} f(x)\, dx - \frac{d}{dx}\left(\int f(x)\, dx\right)$? [2010(A)]

Ans. $\int \frac{d}{dx} f(x) - \frac{d}{dx}\left(\int f(x)\right)$

$= f(x) + C - f(x) = C$ (constant)

21. If $\int_1^2 f(x)\, dx = \lambda$, then what is the value of $\int_1^2 f(3-x)\, dx$? [2010(A)]

Ans. If $\int_1^2 f(x)\, dx = \lambda$, then $\int_1^2 f(3-x)\, dx = \lambda$.

22. What is the value of $\int_{-1}^{1} \frac{dx}{1+x^2}$? [2010(A)]

Ans. $\int_{-1}^{1} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_{-1}^{1}$

$= \tan^{-1} 1 - \tan^{-1} (-1)$

$= \tan^{-1} 1 + \tan^{-1} 1$

$= 2 \tan^{-1} (1) = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$

23. Write the order and the degree of the following differential equation :

$\frac{d^3y}{dx^3} = \left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^4 + y$ [2010(A)]

Ans. Order = 3

Degree = 1

24. Write the particular solution of

$\frac{dy}{dx} = (1+x)^4$, $y = 0$ when $x = -1$. [2010(A)]

Ans. $\frac{dy}{dx} = (1+x)^4$

$\Rightarrow y = \frac{(1+x)^5}{5} + C$

Given $y = 0$ for $x = -1$

$\Rightarrow 0 = 0 + C \Rightarrow C = 0$

$\therefore$ The particular solution is $y = \frac{(1+x)^5}{5}$

25. Write the value of [2009(A)]

$U \int V\, dx - \int U' (\int V\, dx)\, dx - V \int U\, dx + \int V' (\int U\, dx)\, dx$

Ans. $U \int V dx - \int U'(\int V dx)\, dx - V \int U dx + \int V'(\int U dx)\, dx$

$= \int UV dx - \int VU dx$

$= \int 0 dx = C$ (constant)

26. Form the differential equation whose solution is $y = e^{x+a}$. [2009(A)]

Ans. $y = e^{x+a}$ ... (1)

$\Rightarrow \frac{dy}{dx} = e^{x+a}$ ... (2)

$\therefore$ The required diff. equation is $\frac{dy}{dx} = y$.

27. Write the particular solution of $\frac{dy}{dx} = \frac{1}{1+x^2}$, given that $y = 1$, when $x = 0$. [2009(A)]

Ans. $\frac{dy}{dx} = \frac{1}{1+x^2}$

$\Rightarrow \int dy = \int \frac{dx}{1+x^2} + C$

$\Rightarrow y = \tan^{-1} x + C$

Using the condition $y = 1$ when $x = 0$ we have $c = 1$.

$\therefore$ The particular solution is $y = \tan^{-1} x + 1$.

28. Write a primitive of $\sin x + \sec x$. [2008(A)]

Ans. The primitive of $\sin x + \sec x$

$= -\cos x + \ln|\sec x + \tan x| + C$.

29. If f is an even function and $\int_{-2}^{0} f(t)\, dt = \frac{3}{2}$, then find $\int_{-2}^{2} f(x)\, dx$. [2008(A)]

Ans. $\int_{-2}^{2} f(x)\, dx = 2\int_{-2}^{0} f(x)\, dx = 2 \cdot \frac{3}{2} = 3$.

30. Find an antiderivative of $e^x (\tan x + \ln \sec x)$. [2008(A)]

Ans. An antiderivative of $e^x (\tan x + \ln \sec x)$

$= \int e^x (\tan x + \ln (\sec x))\, dx$

$= e^x \ln (\sec x) + C$

$\left(\because \frac{d}{dx} \ln(\sec x) = \frac{1}{\sec x} \cdot \sec x \cdot \tan x = \tan x\right)$

31. Find the order of the differential equation whose general solution is $y = ax^2 + b$, a, b being arbitrary constants. [2008(A)]

Ans. As $y = ax^2 + b$ contains two unknown constants, its differential equation is of order 2.

32. Given the general solution as $y = (x^2 + c)\, e^{-x}$ of a differential equation, what is the particular solution if $y = 0$ when $x = 1$. [2008(A)]

Ans. The general solution is

$$y = (x^2 + c)\, e^{-x} \quad \dots (1)$$

The particular solution if $y = 0$ when $x = 1$

is $y = \frac{(x^2 - 1)}{e^x}$.

($\because$ Putting $x = 1$, $y = 0$ in (1) we have $1 + c = \Rightarrow c = -1$

33. Write the value of $\int_{-\pi/3}^{\pi/3} (x^4 \sin x^3 + x \cos x^2)\,dx$. [2004(A)]

Ans. As $x^4 \sin x^3 + x \cos x^2$ is an odd function.

$$\int_{-\pi/3}^{\pi/3} (x^4 \sin x^3 + x \cos x^2)\,dx = 0.$$

34. What is the area bounded by $x = e^y$, $x = 0$, y [2004(A)]

Ans. Required area $= \int_0^1 x\,dy = \int_0^1 e^y\,dy = \left[e^y\right]_0^1 = e - 1$

35. What is the value of $\int \frac{1 + \frac{1}{x^2}}{x - \frac{1}{x} + 4}\,dx$? [2004(A)]

Ans. *Refer to No. 7 (Level-2) possible questions and answers.*

36. $\int \frac{3dx}{(x-1)(x+2)} =$ —— ? [2004(A)]

Ans. $\int \frac{3dx}{(x-1)(x+2)} = \int \frac{dx}{x-1} - \int \frac{dx}{x+2}$

$$= \ln|x - 1| - \ln(x + 2) + C$$

$$= \ln\left|\frac{x-1}{x+2}\right| + C$$

37. What is the value of $\int (e^x \cos x + e^x \sin x)\,dx$. [2004(A)]

Ans. $\int (e^x \cos x + e^x \sin x)\,dx$

$$= \int e^x \cos x\,dx + \int e^x \sin x\,dx$$

$$= \int e^x \cos x\,dx + e^x \sin x - \int e^x \cos x\,dx + C$$

$$= e^x \sin x + C.$$

38. What is the integrating factor of $\frac{dy}{dx} + C = 0$ [2004(A)]

Ans. The differential equation can be written as

$$\frac{dy}{dx} + 0.y = (-x)$$

Here $P = 0$, $Q = -x$

$\therefore$ The integrating factor $= e^{\int p dr} = e^0 = 1$

39. Evaluate $\int \left(\sqrt{a^2 - x^2} + \frac{x^2}{\sqrt{a^2 - x^2}}\right) dx$. [2003(A)]

Ans. $\int \left(\sqrt{a^2 - x^2} + \frac{x^2}{\sqrt{a^2 - x^2}}\right) dx$

$$= a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} = a^2 \sin^{-1}\frac{x}{a} + C$$

40. Write the differential equation of the parabola $y^2 = 4x + 12$. [2003(A)]

Ans. Given equation is $y^2 = 4x + 12$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4 \quad \Rightarrow y\frac{dy}{dx} = 2$$

$\therefore$ The differential equation is $y\frac{dy}{dx} = 2,\ y(-3) = 0$

41. What is the value of

$$\int_1^3 \tan^{-1} x\,dx + \int_1^3 \cot^{-1} x\,dx \ ?$$ [2002(A)]

Ans. $\int_1^3 \tan^{-1} x\,dx + \int_1^3 \cot^{-1} x\,dx$

$$= \int_1^3 \left\{\tan^{-1} x + \cot^{-1} x\right\} dx = \int_1^3 \frac{\pi}{2}\,dx$$

$$= \frac{\pi}{2}[x]_1^3 = \frac{\pi}{2}(3 - 1) = \frac{\pi}{2} \times 2 = \pi$$

42. Write the value of $\int |x|\,dx$ when $x < 0$. [2001(A)]

Ans. $\int |x|\,dx,\ x < 0 = \int -x\,dx = -\frac{x^2}{2} + C$

43. Write the value of $\int \sin^2 x.d(\sin x)$ [2001(A)]

Ans. $\int \sin^2 x.d(\sin x) = \frac{\sin^3 x}{3} + C$

Differential equations.

$x\,dx + y\,dy = 0$ .... ① order = 1 degree = 1

$\frac{dy}{dx} + 3y = x$ .... ② order = 1 degree = 1

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$ .... ③ order = 2, degree = 1

$\frac{d^n y}{dx^n} + \frac{dy}{dx} = e^{3x}$ ...... ④ order n degree = 1

$\left(\frac{dy}{dx}\right)^2 = x^2 + \sqrt{x}$ .... ⑤ order = 1, degree = 2

$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^5 = x$ ...... ⑥ order = 3, degree 1

$\frac{d^2y}{dx^2} = \sqrt{1 + \frac{dy}{dx}}$ order = 2, degree = 3

$\Rightarrow \left(1 + \frac{d^2y}{dx^2}\right)^2 = 1 + \frac{dy}{dx}$ order, 2 degree, 2.

Formation of Differential equation;

$y = mx$ --- ①
$\hookrightarrow$ constant

$\Rightarrow \frac{dy}{dx} = m$ --- ②

Elemination between ① and ② one get

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$ ...(3)

Exm - 1

Let $y = A\sin x$ ...... ①

To elemenate A we get,

$\frac{dy}{dx} = A\cos x$

$\frac{dy}{dx} = \frac{A\cos x}{\ldots} = \ldots \cot x$

Which differential equation of order one.

Example -2

$Y = Ae^{x} + Be^{-x} \cdots\cdots (1)$

$\frac{dy}{dx} = Ae^{x} - Be^{-x} \cdots\cdots (2)$

And $\frac{dy^2}{dx^2} = Ae^{x} + Be^{-x} \cdots\cdots (3)$

Comparing equation (1) and (2)

$\frac{d^2y}{dx^2} = y$

The equation of order is 2.

Example -3

$y = 3x + c \cdots\cdots (1)$

$\frac{dy}{dx} = 3$

Example -4

METHODS OF SOLVING DIFFERENTIAL EQUATIONS

(i) $\frac{dy}{dx} = f(x) \cdot g(y)$

The equation,

$\frac{dy}{dx} = f(x)$

$\Rightarrow dy = f(x)\,dx$

Integrating in both side equ.

$\Rightarrow \int dy = \int f(x)\,dx$

$\Rightarrow y = \int f(x)\,dx + c$

$\frac{dy}{dx} = g(y) = \int \frac{dy}{g(y)} = x + c$

(ii) $\frac{d^2y}{dx^2} = h(x)$

$\Rightarrow d^2y = h(x)\, dx^2$

$\Rightarrow \int d^2y = h(x)\, dx^2$

$\Rightarrow \int d^2y = h(x) \cdot \frac{x^3}{3}$

Example-5

$$\frac{dy}{dx} = x^2+2x+5$$

$\Rightarrow dy = (x^2+2x+5)\, dx$

Interagrating in bothside

$$\int dy = \int (x^2+2x+5)\, dx$$

$$\Rightarrow y = \frac{x^3}{3} + \frac{2x^2}{2} + 5x$$

$$= \frac{x^3}{3} + x^2 + 5x + C$$

Example-6

$$\frac{dy}{dx} = \tan y$$

$\Rightarrow \int \frac{dy}{dx} = \int \tan y$

$\Rightarrow \frac{y}{x} = \ln \sin y$

$\Rightarrow \ln \sin y = x + C$

Example-7.

$$\frac{dy}{dx} = \frac{2y}{x^2+1} \Rightarrow \frac{dy}{y} = \frac{2dx}{x^2+1}$$

$\Rightarrow \int \frac{dy}{y} = 2\int \frac{dx}{x^2+1} \Rightarrow \ln y = 2\tan^{-1}x + c$

Example - 8

EXERCISE - 1(a)

(1.) (i) $y \sec^2 x\, dx + \tan x\, dy = 0$

$\Rightarrow \frac{dy}{dx}(y\sec^2 x\, dx + \tan x\, dy) = 0$

$\Rightarrow y\tan x + \ln(\sin x) = 0$

Order $= 1$, degree $= 1$

(ii) $\left(\frac{dy}{dx}\right)^4 + y^5 = \frac{d^3y}{dx^3}$

Order $= 3$, degree $= 1$

(iii) $a\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$

$\Rightarrow a\frac{d^2y}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^3$

$\Rightarrow a\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \frac{dy}{dx}$

Order $= 2$, degree $= 2$

(iv) $\tan^{-1}\sqrt{\frac{dy}{dx}} = x$

order $= 1$, degree $= 1$

(v) $\ln\left(\frac{d^2y}{dx^2}\right) = y$

order $= 2$, degree $= 1$

(vi) $\frac{\frac{dy}{dt}}{y + \frac{dy}{dt}} = \frac{yt}{\frac{dy}{dt}} = \left(\frac{d^3y}{dx^3}\right) = \frac{yt}{y}$

(2)(i) $y = A\sec x$

$$\frac{dy}{dx} = A\sec x \cdot \tan x$$

$$= y\tan x$$

(ii) $y = c\tan^{-1}x$

$$\frac{dy}{dx} = c\frac{1}{1+x^2}$$

$$\cancel{\frac{d^2y}{dx^2} = c\frac{1}{1+x^2}\,2x}$$

(iii) $$\int\frac{dy}{dx} = C\int\frac{1}{1+x^2}$$

$$y = Ae^{t} + Be^{2t}$$

$$\Rightarrow \frac{dy}{dx} = Ae^{t}\,\cancel{B2}\,e^{2t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = Ae^{t} + B2e^{2t}$$

$$\frac{d^2y}{dx^2} = y$$

(iv)

$$y = Ax^2 + Bx$$

$$\Rightarrow \frac{dy}{dx} = A\frac{x^3}{3} + B\frac{x^2}{2}$$

$$\frac{d^2y}{dx^2} = \frac{3x^2}{3} + \frac{\cancel{2}x}{\cancel{2}} = x^2 + 2x$$

(ix) $ax^2+by=1$

$\Rightarrow y = -\frac{a}{b}x^2 + \frac{1}{b}$

$\Rightarrow \frac{dy}{dx} = \frac{2a}{b}x \Rightarrow \frac{\frac{dy}{dx}}{x} = \frac{-2a}{b}$

Again differentiating we get,

$$\frac{d^2y}{dx^2}\cdot x - \frac{dy}{dx} = 0$$

(3) (i) $\frac{dy}{dx} = \frac{e^{2x}+1}{e^x}$

$\Rightarrow \int dy = \int (e^x + e^{-x})\,dx$

$\Rightarrow y = e^x - e^{-x} + c$

(ii) $\frac{dy}{dx} = x\cos x$

$\Rightarrow \int dy = \int x\cos x\,dx$

$\Rightarrow y = \cancel{x\sin x + \cos x} + C$

$\int (\cos x\,dx)\,x - \int \sin x \cdot 1$

$= \sin x \cdot 1 - (-\cos x)$

$= \sin x + \cos x + C$

$x^2-7x+12$

$=(x^2-4x)-3x+12$

$= x(x-4)-3(x-4)$

$(x-4)(x-3)$

(iii) $\frac{dy}{dt} = t^5 \log t$

$\Rightarrow \int dy = \int \ln t \cdot t^5\,dt$

$\Rightarrow y = \int (t^5\,dx)\ln t - \int \left[\int (t^5 dx) \cdot \frac{d}{dx}\ln t \right] dx$

(iv) $\frac{dy}{dt} = 3t^2 + 4t + \sec^2 t$

$\Rightarrow dy = (3t^2 + 4t + \sec^2 t)\, dt$

Integrating the above eqⁿ in both side

$\int dy = \int (3t^2 + 4t + \sec^2 t)\, dt$

$= 3\int t^2 + 4\int t + \int \sec^2 t\, dt$

$= \cancel{3}\frac{t^3}{\cancel{3}} + \cancel{4}^{2}\frac{t^2}{\cancel{2}} + \tan t + c$

$= t^3 + 2t^2 + \tan t + c$

(v) $\frac{dy}{dx} = \frac{1}{x^2 - 7x + 12}$

$\Rightarrow dy = \frac{dx}{x^2 - 7x + 12}$

Integrating the above eqⁿ in both side,

$\int dy = \int \frac{dx}{x^2 - 7x + 12}$

$\Rightarrow y = \int \frac{dx}{(x^2 - 4x) - 3x + 12}$

$= \int \frac{dx}{x(x-4) - 3(x-4)}$

$= \int \frac{dx}{(x-4)(x-3)}$

$= \int \left(\frac{1}{x-4} - \frac{1}{x-3} \right) dx$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-1}{v+1}-v=\frac{v-1-v^2-v}{v+1}=\frac{-(1+v^2)}{v+1}$$

$$\Rightarrow \int\frac{v+1}{1+v^2}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \int\frac{vdv}{1+v^2}=\int\frac{dv}{1+v^2}=-\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}ln(1+v^2)+\tan^{-1}v=-ln\,x+C$$

$$\Rightarrow ln\left(\sqrt{1+\frac{y^2}{x^2}}\cdot x\right)+\tan^{-1}\frac{y}{x}=C$$

$$\Rightarrow ln\sqrt{x^2+y^2}+\tan^{-1}\frac{y}{x}=C$$ is the solution of the given differential equation.

30. Find the differential equation whose general solution is $ax^2 + by = 1$, where a and b are arbitrary constants. **[2014 (A)**

Ans. $ax^2+by=1 \Rightarrow 2ax+b\frac{dy}{dx}=0$

Differentiating w.r.t. x we get

$$\frac{2a}{6}+\frac{1}{x}\frac{dy}{dx}=0$$

Differentiating w.r.t. x we get

$$-\frac{1}{x^2}\frac{dy}{dx}+\frac{1}{x}\frac{d^2y}{dx^2}=0$$

$$\Rightarrow x\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$$ is the required differential equation.

31. Integrate :

$$\int\frac{\sin 6x+\sin 4x}{\cos 6x+\cos 4x}dx.$$ **[2013 (A)**

Ans. $$\int\frac{\sin 6x+\sin 4x}{\cos 6x+\cos 4x}dx$$

$$=\int\frac{2\sin 5x\cdot\cos x}{2\cos 5x\cdot\cos x}dx$$

$$=\int\tan 5x\,dx=\frac{1}{5}ln|\sec 5x|+C$$

32. Integrate : $\int\frac{dx}{x\{(\log x)^2+25\}}$ **[2013 (A)**

Ans. $I=\int\frac{dx}{x\{(\log x)^2+25\}}$

Let $\log x = t$,

$$\Rightarrow \frac{1}{x}dx=dt$$

$$\therefore I=\int\frac{dt}{t^2+5^2}=\frac{1}{5}\tan^{-1}\frac{t}{5}+C.$$

[2013 (A)

33. Integrate : $\int\sin^{-1}x\,dx.$

Ans. $I=\int\sin^{-1}x\,dx=\int\sin^{-1}\cdot dx$

$$=(\sin^{-1}x)x-\int\frac{x}{\sqrt{1-x^2}}dx$$

Let us put $1-x^2=t^2$

$\Rightarrow -2x\,dx=2tdt$

$\Rightarrow x dx = -tdt$

$$\therefore \int\frac{xdx}{\sqrt{1-x^2}}=-\int\frac{tdt}{t}=-t+C$$

$$=-\sqrt{1-x^2}+C$$

$$\therefore I=x\sin^{-1}x+\sqrt{1-x^2}+C$$

34. Evaluate :

$$\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx.$$ **[2013 (A)**

$$\textbf{Ans.} I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$=\int_0^{\pi/2}\frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}dx$$

$$=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

$$\therefore 2I=I+I$$

$$=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx++\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$=\int_0^{\pi/2}dx=[x]_0^{\pi/2}=\pi/2$$

$$\therefore I=\frac{\pi}{4}$$

35. Solve : $\frac{dy}{dt}=e^{2t+3y}$ **[2013 (A)**

Ans. $\frac{dy}{dt}=e^{2t+3y}=e^{2t}\cdot e^{3y}$

$$\Rightarrow \int\frac{dy}{e^3y}=\int e^{2t}dt$$

$$\Rightarrow \int e^{-3y}dy=\int e^{2t}dt$$

$$\Rightarrow \frac{e^{-3y}}{-3} = \frac{e^{2t}}{2} + C_1$$

$\Rightarrow 2e^{-3y} + 3e^{2t} = C$ is the general solution.

36. Solve : $\dfrac{dy}{dx} + y = e^{-x}$ **[2013 (A)]**

Ans. $\dfrac{dy}{dx} + y = e^{-x}$

which is a linear differential equation with $P = 1$ and $Q = e^{-x}$

$\therefore$ Integrating factor $= e^{\int p dx} = e^{\int dx} = e^x$

$\therefore$ The solution is

$$y \cdot e^x = \int e^x \cdot e^{-x} dx$$

$\Rightarrow ye^x = x + C$

37. Find the differential equation whose general solution is $y = a \cos x + b \sin x$. **[2013(A), 2007(A)]**

Ans. Given general solution is

$y = a \cos x + b \sin x$ ..... (1)

$\Rightarrow \dfrac{dy}{dx} = -a \sin x + b \cos x$ ..... (2)

$\Rightarrow \dfrac{d^2y}{dx^2} = -a \cos x - b \sin x = -y$

$\therefore$ The required differential equation is

$$\frac{d^2y}{dx^2} + y = 0$$

38. Integrate : $\displaystyle\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\, dx$ ***[2012(A)]***

Ans. $\displaystyle I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\, dx$

Let $x = \cos 2\theta$

$\Rightarrow dx = -2 \sin 2\theta d\theta$

$$\therefore \sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} = \tan\theta$$

$$\therefore I = \int \tan^{-1}(\tan\theta)(-2\sin 2\theta)\, d\theta$$

$$= -2\int \theta \sin 2\theta d\theta$$

$$= (-2)\left[\theta\left(-\frac{\cos 2\theta}{2}\right) - \int 1\cdot\left(-\frac{\cos 2\theta}{2}\right)d\theta\right]$$

$$= \theta\cos 2\theta - \int \cos 2\theta d\theta$$

$$= \theta\cos 2\theta - \frac{\sin 2\theta}{2} + C$$

$$= \frac{1}{2}x\cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$$

39. Evaluate :

$$\int_{-\frac{3}{5}}^{\frac{3}{5}} [2x+1]\, dx$$ ***[2012(A)]***

Ans. When $-\dfrac{3}{5} < x < \dfrac{3}{5}$

We have $-\dfrac{6}{5} < 2x < \dfrac{6}{5}$

$$\Rightarrow -\frac{6}{5}+1 < 2x+1 < \frac{6}{5}+1$$

$$\Rightarrow -\frac{1}{5} < 2x+1 < \frac{11}{5}$$

$$\text{Now } [2x+1] = \begin{cases} -1, & -\frac{1}{5} < 2x+1 < 0 \\ 0, & 0 \le 2x+1 < 1 \\ 1, & 1 \le 2x+1 < 2 \\ 0, & 2 \le 2x+1 < \frac{11}{5} \end{cases}$$

$$= \begin{cases} -1, & -\frac{3}{5} < x < -\frac{1}{2} \\ 0, & -\frac{1}{2} \le x < 0 \\ 1, & 0 \le x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x < \frac{3}{5} \end{cases}$$

$$\therefore \int_{-3/5}^{3/5} [2x+1]\, dx$$

$$= \int_{-3/5}^{-1/2} [2x+1]\, dx + \int_{-1/2}^{0} [2x+1]\, dx + \int_{0}^{\frac{1}{2}} [2x+1]\, dx + \int_{1/2}^{3/5} [2x$$

$$= \int_{-3/5}^{-1/2} [-1]\, dx + \int_{-1/2}^{0} 0\, dx + \int_{0}^{1/2} dx + \int_{1/2}$$

$$= -[x]_{-3/5}^{-1/2} + [x]_0^{1/2} + 2[x]_{1/2}^{3/5}$$

$$= -\left(-\frac{1}{2}+\frac{3}{5}\right) + \left(\frac{1}{2}-0\right) + 2\left(\frac{3}{5}\right.$$

$$= -\frac{1}{10} + \frac{1}{2} + \frac{2}{10} = \frac{1}{10} + \frac{1}{2} =$$

43. Integrate : $\int \frac{(e^x-1)dx}{e^x+1}$ **[2011(A)]**

Ans. $\int \frac{e^x-1}{e^x+1}dx$

$= \int \frac{e^x dx}{e^x+1} - \int \frac{dx}{e^x+1}$

$= \int \frac{e^x dx}{e^x+1} - \int \frac{e^x dx}{e^x(e^x+1)}$

$= \int \frac{dt}{t+1} - \int \frac{dt}{t(t+1)}$ (Where $e^x = t$)

$= \int \frac{dt}{t+1} - \int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt$

$= 2\int \frac{dt}{t+1} - \int \frac{dt}{t}$

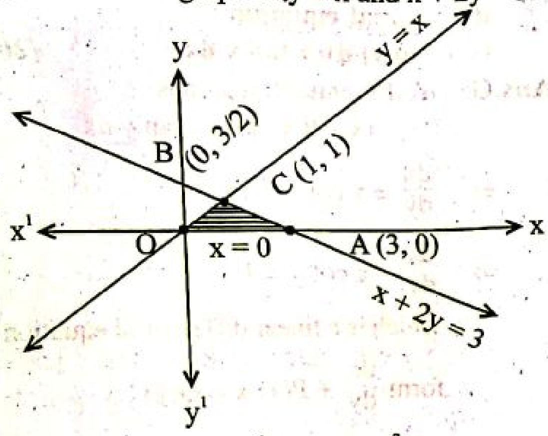
$= 2 ln|t + 1| - ln\ t| + C$

$= 2ln\ (e^x + 1) - ln e^x + C$

$= 2ln(e^x + 1) - x + C.$

44. Find by integration the area bounded by the straight lines y = 0, y = x and x + 2y = 3. **[2011(A)]**

Ans. Let us draw the graph of $y = x$ and $x + 2y = 3$.



The required area $= \int_{0\,[\text{for } y=x]}^{1} y dx + \int_{1\,[\text{for } x+2y=3]}^{3} y dx$

$= \int_0^1 x dx + \int_1^3 \frac{3-x}{2} dx$

$= \left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_1^3$

$= \left(\frac{1}{2} - 0\right) + \frac{1}{2}\left[\left(9 - \frac{9}{2}\right) - \left(3 - \frac{1}{2}\right)\right]$

$= \frac{1}{2} + \frac{1}{2}[6 - 5]$

$= 1$ sq. units.

45. Evaluate : $\int_0^4 [\sqrt{x}]dx$ **[2011(A)]**

Ans. Clearly $[\sqrt{x}] = \begin{cases} 0, 0 < x < 1 \\ 1, 1 \le x < 4 \end{cases}$

$\therefore \int_0^4 [\sqrt{x}]dx = \int_0^1 0 dx + \int_1^4 dx = [x]_1^4 = 3$

46. Write an integrating factor of the following differential equation :

$(x - ln\ y)\frac{dy}{dx} = -y\ ln\ y.$ **[2011(A)]**

Ans. $(x - ln\ y)\frac{dy}{dx} = -y\ ln\ y.$

$\Rightarrow \frac{dx}{dy} = \frac{ln\ y - x}{y ln\ y} = \frac{1}{y} - \frac{x}{y ln\ y}$

$\Rightarrow \frac{dx}{dy} + \frac{x}{y ln\ y} = \frac{1}{y}$

Which linear in x

The integrating factor $= e^{\int \frac{dy}{y ln\ y}}$

$= e^{ln(ln\ y)} = ln\ y$

47. Solve : $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ **[2011(A)]**

Ans. $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$

Which is a homogeneous differential equation

putting $y = vx$ i.e $\frac{dy}{dx} = v + \frac{xdv}{dx}$

We get $v + x\frac{dv}{dx} = \frac{v^2x^2}{vx^2 - x^2} = \frac{v^2}{v-1}$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1} = \frac{v}{v-1}$

$\Rightarrow \int \frac{(v-1)dv}{v} = \int \frac{dx}{x}$

$\Rightarrow \int \left(1 - \frac{1}{v}\right)dv = \int \frac{dx}{x}$

$\Rightarrow v - ln\ v = ln\ x + C$

$\Rightarrow v = ln\ vx + C$

$\Rightarrow \frac{y}{x} = ln\ y + C$

$\Rightarrow y = x(ln\ y + C)$ is the required solution.

48. Integrate :

$\int \sin^4 x \cos^3 x\, dx$ **[2010(A)]**

Ans. $I = \int \sin^4 x \cos^3 x\, dx$

$$= \int \sin^4 x(1-\sin^2 x)\cos x\,dx$$

$$= \int (t^4 - t^6)dt \quad \text{(where } \sin x = t)$$

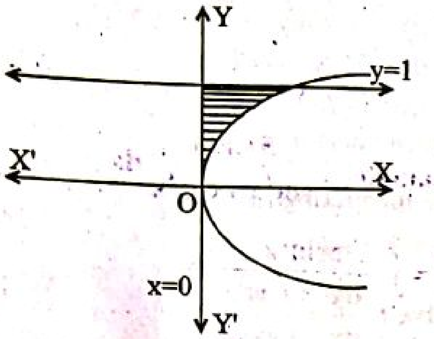
$\Rightarrow \cos x\,dx = dt)$

$$= \frac{t^5}{5} - \frac{t^7}{7} + C$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C'$$

49. Find the area enclosed by the curve $y^2 = x$, and the straight lines $x = 0$, $y = 1$. ***[2010(A)]***

Ans. Given curves is $y^2 = x$, which is a parabola with vertex at (0, 0), symmetrical about x-axis and open to right.



We want to find the area of shaded region.

$$\text{Area} = \int_0^1 x\,dy = \int_0^1 y^2\,dy = \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3} \text{ sq units}$$

50. Find the differential equation whose general solution is $c_1x^2 + c_2y = 1$ where c_1, c_2 are arbitrary constants. ***[2010(A)]***

Ans. Given solution is : $c_1x^2 + c_2y = 1$ ... (1)

From (1) we have

$$2c_1x + c_2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2c_1x}{c_2} \quad \text{... (2)}$$

$$\text{Again } \frac{d^2y}{dx^2} = \frac{-24}{c_2} \quad \text{... (3)}$$

From Eq^n (2) and (3) we have

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{dy}{dx}$$

$\Rightarrow x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0$ is the required differential equation.

51. Find the particular solution of the following differential equation : ***[2010(A)]***

$$\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0, \; y(-1) = -\sqrt{3}$$

Ans. Given differential equation is

$$\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0, \; y(-1) = -\sqrt{3}$$

$$\Rightarrow \int \frac{dy}{1+y^2} + \int \frac{dx}{1+x^2} = c$$

$$\Rightarrow \tan^{-1} y + \tan^{-1} x = c$$

Putting $x = -1$ and $y = -\sqrt{3}$ we have

$$\tan^{-1}(-\sqrt{3}) + \tan^{-1}(-1) = c$$

$$\Rightarrow -\frac{\pi}{3} - \frac{\pi}{4} = c$$

$$\Rightarrow c = -\frac{7\pi}{12}$$

$\therefore$ The required particular solution is

$$\tan^{-1} x + \tan^{-1} y + \frac{7\pi}{12} = 0.$$

52. Find an integrating factor of the following differential equation : ***[2010(A)]***

$(x + \tan y)\,dy = \tan y\,dx$

Ans. Given differential equation is

$$(x + \tan y)\,dy = \tan y\,dx$$

$$\Rightarrow \frac{dx}{dy} = x\cot y + 1$$

$$\Rightarrow \frac{dx}{dy} - x\cot y = 1$$

which is a linear differential equation of the form $\dfrac{dx}{dy} + P(y)\,x = Q(y)$

$\therefore$ Integrating factor $= e^{\int P dy}$

$$= e^{-\int \cot y\,dy} = e^{\ln \operatorname{cosec} y} = \operatorname{cosec} y.$$

53. Integrate :

$$\int \frac{\sec x \operatorname{cosec} x}{\ln \tan x}dx \quad \textit{[2009(A)]}$$

$$\text{Ans. } I = \int \frac{\sec x \operatorname{cosec} x}{\ln \tan x}dx$$

Let us put $\ln \tan x = t$

$$\Rightarrow \frac{1}{\tan x}\cdot \sec^2 x\,dx = dt$$

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65. $\int \sin x \cos x\, dx = \int \sin x\, d(\sin x) = \frac{\sin^2 x}{2} + C_1$

$\int \sin x \cos x\, dx = -\int \cos x\, d(\cos x) = \frac{\cos^2 x}{2} + C_2$

where C_1 and C_2 are constants. Explain the double answer of the same integral. *[2005(A)]*

Ans. Any two indefinite integrals of a function differ by a constant.

Hence $\left(\frac{1}{2}\sin^2 x + C_1\right) - \left(-\frac{1}{2}\cos^2 x + \frac{C}{2}\right)$

$= \frac{1}{2} + C_1 - C_2 = C$ (say) a constant.

So both the answers are correct.

66. **Evaluate :** $\int_0^{1.415} [x^2] dx$ *[2005(A)]*

Ans. $\int_0^{1.415} [x^2] dx$

$= \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{1.415} [x^2] dx$

$= \int_0^1 0.dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{1.415} 2.dx$

$= 0 + [x]_{\sqrt{2}}^1 + 2[x]_{\sqrt{2}}^{1.415}$

$= 0 \pm \sqrt{2} - 1 + 2(1.415 - \sqrt{2})$

$= \sqrt{2} - 1 + 2.830 - 2\sqrt{2} = 1.830 - \sqrt{2}$

67. **$f'(x) - g''(x) = 0$, $f(1) = 3$, $g(1) = 2$, $f'(0) = 2$ and $g'(0) = 1$, then find $f(x)$.** *[2005(A)]*

Ans. Given $f''(x) - g''(x) = 0$.

$f(1) = 3$, $g(1) = 2$, $f'(0) = 2$ and $g'(0) = 1$

Now $f''(x) - g''(x) = 0$

Integrating, $f'(x) - g'(x) = C_1$ ......(i)

Again integrating, $f(x) - g(x) = C_1 x + C_2 x$.(ii)

Using given values in equation (i) and (ii) we get $C_1 = 1$ and $C_2 = 0$.

Putting in (ii),

$f(x) - g(x) = x + 0 \Rightarrow f(x) = x + g(x)$

68. **Evaluate** $\int_{-1}^{1} (|x| + x)^2 dx$ *[2005(A)]*

Ans. $\int_{-1}^{1} (|x| + x)^2 dx$

$= \int_{-1}^{0} (|x| + x)^2 .dx + \int_0^1 (|x| + x)^2 dx$

$= \int_{-1}^{0} (-x + x)^2 .dx + \int_0^1 (x + x)^2 dx$

$= \int_{-1}^{0} 0.dx + \int_0^1 (2x)^2 dx = 0 + \int_0^1 4x^2 .dx$

$= \left[4.\frac{x^3}{3}\right]_0^1 = \frac{4}{3}(1 - 0) = \frac{4}{3}$

69. **Integrate** $\int \frac{dx}{\sqrt{5 + 4x - x^2}}$ *[2004(A)]*

Ans. $\int \frac{dx}{\sqrt{5 + 4x - x^2}}$

$= \int \frac{dx}{\sqrt{9 - (x-2)^2}} = \int \frac{dx}{\sqrt{3^2 - (x-2)^2}}$

$= \sin^{-1} \frac{x-2}{3} + C$

70. **Integrate :** $\int \frac{4x - 5}{x^2 - x - 2} dx$ *[2004(A)]*

Ans. $\int \frac{4x - 5}{x^2 - x - 2} dx = \frac{1}{2} \int \frac{2x - \frac{5}{2}}{x^2 - x - 2} dx$

$= \int \frac{(2x - 1) - \frac{3}{2}}{x^2 - x - 2} dx$

$= \int \frac{2x - 1}{x^2 - x - 2} dx - \frac{3}{2} \int \frac{dx}{x^2 - x - 2}$

$= \ln|x^2 - x - 2| - \frac{3}{2} \int \frac{dx}{x^2 - 2x.\frac{1}{2} + \frac{1}{4} - \frac{9}{4}}$

$= \ln|x^2 - x - 2| - \frac{3}{2} \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$

$= \ln|x^2 - x - 2| - \frac{3}{2}.\frac{1}{2 \times \frac{3}{2}} \ln\left|\frac{x - \frac{1}{2} - \frac{3}{2}}{x - \frac{1}{2} + \frac{3}{2}}\right|$

$= \ln|x^2 - x - 2| - \frac{1}{2} \ln\left|\frac{x-2}{x+1}\right| + C$

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$$\Rightarrow \int e^{3x}\sin 4x\,dx = \frac{e^{3x}}{25}(3\sin 4x - 4\cos 4x) + K$$

where $K = \frac{9C}{25}$

18. Solve the following differential equation :
$(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$ *[2010(A)*

Ans. $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{(2x+y+1)}{4x+2y-1}$$

Let $2x + y = z$

$$\Rightarrow 2 + \frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dz}{dx} - 2 = \frac{-(z+1)}{2z-1}$$

$$\therefore \frac{dz}{dx} = 2 - \frac{z+1}{2z-1} = \frac{4z-2-z-1}{2z-1} = \frac{3(z-1)}{2z-1}$$

$$\Rightarrow \int \frac{(2z-1)}{z-1}dz = \int 3dx$$

$$\Rightarrow \int \frac{2z-2+1}{z-1}dz = 3\int dx$$

$$\Rightarrow \int\left(2 + \frac{1}{z-1}\right)dz = 3\int dx$$

$$\Rightarrow 2z + \ln|z-1| = 3x + C$$

$$\Rightarrow 2(2x+y) + \ln|2x+y-1| = 3x + C$$

$$\Rightarrow x + 2y + \ln|2x+y-1| = C$$

19. Integrate :

$$\int \frac{\cos x}{\sin 2x + \sin x}dx \qquad \textit{[2009(A)]}$$

Ans. $I = \int \frac{\cos x}{\sin 2x + \sin x}$

$$= \int \frac{\cos x}{\sin x(2\cos x + 1)}$$

$$= \int \frac{\cos x \sin dx}{\sin^2 x(2\cos x + 1)}$$

$$= \int \frac{\cos x \sin dx}{(1-\cos^2 x)(2\cos x + 1)}$$

Let $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$\therefore I = -\int \frac{t\,dt}{(1-t^2)(2t+1)}$$

$$= \int \frac{t\,dt}{(t+1)(t-1)(2t+1)}$$

Let $\frac{t}{(t+1)(t-1)(2t+1)} = \frac{A}{t+1} + \frac{B}{t-1} + \frac{C}{2t+1}$

$$= \frac{A(1-1)(2t+1) + B(t+1)(2t+1) + C(t+1)(t-1)}{(t+1)(t-1)(2t+1)}$$

$\Rightarrow$ $t = A(t-1)(2\ldots$ is an identity
$+ C(t+1)(t-1)$

Let $t = 1 \Rightarrow 1 = B(2)(3) \Rightarrow B = \frac{1}{6}$

Let $t = -1 \Rightarrow A(-2)(-1) \Rightarrow A = \frac{-1}{2}$

$t = 0 \Rightarrow 0 = A(-1) + B(1) + C(-1)$

$$\Rightarrow C = B - A = \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore I = \left(\frac{-1}{2}\right)\int \frac{dt}{t+1} + \frac{1}{6}\int \frac{dt}{t-1} + \frac{2}{3}\int \frac{dt}{2t+1}$$

$$= \left(\frac{-1}{2}\right)\ln|t+1| + \frac{1}{6}\ln|t-1| + \frac{2}{3}\cdot\frac{1}{2}\ln|2t+1| + C$$

$$= \left(\frac{-1}{2}\right)\ln|\cos x + 1| + \frac{1}{6}\ln|\cos x - 1|$$

$$+ \frac{1}{3}\ln|2\cos x + 1| + C$$

20. Solve :

$$\frac{d^2y}{dx^2} = \frac{1}{x(x+1)} + \operatorname{cosec}^2 x \qquad \textit{[2009(A)]}$$

Ans. Given differential equation is

$$\frac{d^2y}{dx^2} = \frac{1}{x(x+1)} + \operatorname{cosec}^2 x$$

$$= \frac{1}{x} - \frac{1}{x+1} + \operatorname{cosec}^2 x$$

$$\Rightarrow \frac{dy}{dx} = \int\left(\frac{1}{x} - \frac{1}{x+1} + \operatorname{cosec}^2 x\right)dx + C_1$$

$$= \ln|x| - \ln|x+1| - \cot x + C_1$$

$$\Rightarrow y = \int(\ln|x| - \ln|x+1| - \cot x\,dx + C_2$$

Now $I_1 = \int \ln x\,dx = x\ln x - \int dx = x\ln x - x$

$$\therefore y = x\ln|x| - x - \{(x+1)\ln|x-1| - (x)\}$$
$$- \ln|\sin x| + C_1 x + C_2$$

$$= x[\ln|x| - \ln|x+1|] - \ln|x+1|$$
$$- \ln|\sin x| + C_1 x + C_2$$

$$= x\ln\left|\frac{x}{x+1}\right| - \ln|x+1| - \ln|\sin x|$$
$$+ C_1 x + C_3 \text{ (where } C_3 = C_2 + 1)$$

21. Integrate :

$$\int \frac{2x+1}{\sqrt{x^2+10x+29}}dx \qquad \textit{[2008(A)]}$$

GROUP - A

1. **If $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2=144$, write the value of ab.** [2018(A)]

Ans. $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2=144$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta=144$

$\Rightarrow a^2b^2(\sin^2\theta+\cos^2\theta)=144$ $\left\{\begin{array}{l}|\vec{a}|=a\\ |\vec{b}|=b\end{array}\right.$

$\Rightarrow (ab)^2=144$

$\Rightarrow ab=\sqrt{144}=12$

2. **Write the equations of the line $2x+z-4=0$ $=2y+z$ in the symmetrical form.** [2018(A)]

Ans. Given line is $2x+z-4=0=2y+z$

$\Rightarrow z=-(2x-4)=-2(x-2)$ and $z=-2y$

$\therefore -2(x-2)=-2y=z$

$\therefore$ The equation of the line in symmetrical form is $\dfrac{x-2}{1}=\dfrac{y}{1}=\dfrac{z}{-2}$.

3. **If the vectors $\vec{a},\vec{b}$ and $\vec{c}$ from the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively of a triangle ABC, then write the value of $\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.** [2017(A)]

Ans. Clearly $\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{BA}$

$\Rightarrow \vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow (\vec{a}+\vec{b})\times\vec{c}=-\vec{c}\times\vec{c}=0$

$\Rightarrow \vec{a}\times\vec{c}+\vec{b}\times\vec{c}=0$

4. **If $|\vec{a}|=3, |\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=0$, then write the value of $|\vec{a}\times\vec{b}|$.** [2016 (A)]

Ans. As $\vec{a}.\vec{b}=0$ and $|\vec{a}|\neq 0, |\vec{b}|\neq 0$

We have $\cos\theta=0\Rightarrow\theta=\dfrac{\pi}{2}$

$\therefore |\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta=3\times2\times\sin\dfrac{\pi}{2}=6$

5. **Write the distance between parallel planes $2x-y+3z=4$ and $2x-y+3z=18$** [2016 (A)]

Ans. Distance between the given parallel planes

$=\dfrac{|18-4|}{\sqrt{4+1+9}}=\dfrac{14}{\sqrt{14}}=\sqrt{14}$

6. **Write the unit vectors in R^3 which makes angles 45° and 60° with positive directions of x-axis and y-axis respectively.** [2015 (A)]

Ans. Given $\alpha=45^\circ, \beta=60^\circ$

Now $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\Rightarrow \cos^2(45^\circ)+\cos^2(60^\circ)+\cos^2\gamma=1$

$\Rightarrow \dfrac{1}{2}+\dfrac{1}{4}+\cos^2\gamma=1$

$\Rightarrow \cos^2\gamma=1-\dfrac{3}{4}=\dfrac{1}{4}\Rightarrow\cos\gamma=\pm\dfrac{1}{2}$

$\therefore$ The required unit vector is

$\hat{a}=(\cos\alpha)\hat{i}+(\cos\beta)\hat{j}+(\cos\gamma)\hat{k}$

$=\dfrac{1}{\sqrt{2}}\hat{i}+\dfrac{1}{2}\hat{j}\pm\dfrac{1}{2}\hat{k}$

7. **Write down the equation to the plane perpendicular to the y-axis at the point (0, –2, 0).** [2015 (A)]

Ans. The equation of the plane perpendicular to y-axis at (0, –2, 0) is

$(x-0)\cdot0+(y+2)\cdot1+(z-0)\cdot0=0$

$\Rightarrow y+2=0$

8. **Under which conditions the straight line $\dfrac{x-a}{l}=\dfrac{y-b}{m}=\dfrac{z-c}{n}$ intersects the plane $Ax+By+Cz=0$ at a point other than (a, b, c) ?** [2014 (A)]

Ans. The line $\dfrac{x-a}{l}=\dfrac{y-b}{m}=\dfrac{z-c}{n}$ will intersec the plane $Ax+By+Cz+D=0$ at a point othe than (a, b, c) if $Al+Bm+Cn\neq0$ and $Aa+B$ $+Cc+D\neq0$.

9. **How many straight lines in space throu the origin are equally inclined to the ordinate axes ?** [2014

Ans. There are two lines in space through or which are equally inclined to co-ordinate ax

10. Write the value of α if the the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{b} = \alpha\hat{i} - \hat{j} + 2\hat{k}$ are parallel. [2013 (A)

Ans. $\vec{a} = 2i + 3j - 6x$ and $\vec{b} = \alpha i - j + 2k$ are parallel if $\frac{\alpha}{2} = \frac{-1}{3} = \frac{2}{-6} \Rightarrow \alpha = \frac{-2}{3}$

11. Write the equation of the line passing through the point (4, –6, 1) and parallel to the line $\frac{x-1}{1} = \frac{y+2}{3} = \frac{z-1}{-1}$ [2013 (A)

Ans. Equation of the line passing through (4,–6, 1) and parallel to $\frac{x-1}{1} = \frac{y+2}{3} = \frac{z-1}{-1}$ is $\frac{x-4}{1} = \frac{y+6}{3} = \frac{z-1}{-1}$

12. What is the image of the point (–2, 3, –5) with respect to the zx-plane ? [2013 (A)

Ans. Image of the point (–2, 3, –5) w.r.t. zx – plane is (–2, –3, –5).

13. What is the point of intersection of the line x =y = z with the plane x + 2y + 3z = 6 ? [*2012(A)*

Ans. Given line is $x = y = z = \lambda$ (say)

Any point on this line has co-ordinates $(\lambda, \lambda, \lambda)$.

Putting $x = y = z = \lambda$ in $x + 2y + 3z = 6$

We get $6\lambda = 6 \Rightarrow \lambda = 1$

$\therefore$ The point of interesction is (1, 1, 1).

14. Write the component of the vector $\vec{b} = 8\hat{i} + \hat{j}$ in the direction of the vector $\vec{a} = \hat{i} + 2\hat{j} - 2\hat{k}$. [*2012(A)*

Ans. The component of $\vec{b} = 8\hat{i} + \hat{j}$ in the direction of $\vec{a} = \hat{i} + 2\hat{j} - 2\hat{k}$.

$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

$$= \frac{(8\hat{i} + \hat{j}).(\hat{i} + 2\hat{j} - 2\hat{k})}{(1+4+4)} (\hat{i} + 2\hat{j} - 2\hat{k})$$

$$= \frac{10}{9} (\hat{i} + 2\hat{j} - 2\hat{k})$$

15. To which coordinate axis is the plane 2x + 3z = 0 parallel and why ? [*2012(A, 2009(A)*

Ans. The plane 2x + 3z = 0 is parallel to y-axis. Because angle θ between the plane and line is given by

$$\sin\theta = \frac{2 \cdot 0 + 0 \cdot 1 + 3 \cdot 0}{\sqrt{13}\sqrt{1}}$$

$\Rightarrow \sin\theta = 0$

$\Rightarrow \theta = 0$

16. Write the value of m and n for which the vectors $(m-1)\hat{i} + (n+2)\hat{j} + 4\hat{k}$ and $(m+1)\hat{i} + (n-2)\hat{j} + 8\hat{k}$ will be parallel. [*2011 (A)*

Ans. $(m-1)\hat{i} + (n+2)\hat{j} + 4\hat{k}$ and $(m+1)\hat{i} + (n-2)\hat{j} + 8\hat{k}$ are parallel if

$$\frac{m-1}{m+1} = \frac{n+2}{n-2} = \frac{4}{8} = \frac{1}{2}$$

$\Rightarrow 2m - 2 = m + 1$ and $2n + 4 = n - 2$

$\Rightarrow m = 3$ and $n = -6$.

17. Write vector normal to $(\hat{i}+\hat{k})$ and $(\hat{i}+\hat{j})$. [*2011(A), 2003(A)*

Ans. A vector normal to the given vectors is

$$= (\hat{i}+\hat{k}) \times (\hat{i}+\hat{j})$$

$$= \hat{i}\times\hat{i} + \hat{i}\times\hat{j} + \hat{k}\times\hat{i} + \hat{k}\times\hat{j}$$

$$= \hat{k} + \hat{j} - \hat{i} = -\hat{i} + \hat{j} + \hat{k}$$

18. How many directions a null vector has ? [*2010(A)*

Ans. A null vector has infinitely many directions (arbitrary direction).

19. For what value of λ the vectors $\lambda\hat{i} + 3\hat{j} + \lambda\hat{k}$ and $\lambda\hat{i} - 2\hat{j} + \hat{k}$ are perpendicular to each other. [*2010(A)*

Ans. Given vectors are perpendicular to each other iff $\lambda^2 - 6 + \lambda = 0$.

$\Rightarrow \lambda^2 + 3\lambda - 2\lambda - 6 = 0$

$\Rightarrow \lambda(\lambda + 3) - 2(\lambda + 3) = 0$

$\Rightarrow (\lambda - 2)(\lambda + 3) = 0$

$\Rightarrow \lambda = 2, \lambda = (-3)$

20. ... $= 2$ and $|z| = 3$, then how many points in R^3 are there having co-ordinates (x, y, z)? [2010(A)]

Ans. Required number of points are '8'.

21. Write the equation of the plane passing through the point (1, –2, 3) and perpendicular to the y-axis. [2010(A)]

Ans. D.rs. of any line parallel to y-axis are <0, 1, 0>.

$\therefore$ The equation of required plane is

$(x-1)\cdot 0 + (y+2)\cdot 1 + (z-3)\cdot 0 = 0$

$\Rightarrow y + 2 = 0$

22. Determine μ for which the vector $\vec{\alpha} = \mu(6\hat{i} + 2\hat{j} - 3\hat{k})$ will be of unit length. [2009(A)]

Ans. $|\vec{\alpha}| = 1$

$\Rightarrow |\mu\sqrt{36+4+9}| = 1$

$\Rightarrow \mu = \pm\frac{1}{7}$

23. Write the values of a and b for which the vectors $(a-1)\hat{i} + (b+2)\hat{j} + 4\hat{k}$ and $(a+1)\hat{i} + (b-2)\hat{j} + 8\hat{k}$ will be parallel. [2009(A)]

Ans. Given vectors are parallel iff

$$\frac{a-1}{a+1} = \frac{b+2}{b-2} = \frac{4}{8} = \frac{1}{2}$$

$\Rightarrow 2a - 2 = a + 1$

and $2b + 4 = b - 2$

$\Rightarrow a = 3$ and $b = -6$

24. If A, B, C, D, E are the vertices of a regular pentagon, find the vector sum $\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EA}$. [2008(A)]

Ans. $\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EA} = 0$

25. If $\vec{a}\cdot\vec{b} = 0$ and $\vec{a}\times\vec{b} = \vec{0}$, then draw the conclusion. [2008(A)]

Ans. If $\vec{a}\times\vec{b} = 0$ and $\vec{a}\cdot\vec{b} = \vec{0}$, then either or both of $\vec{a}$ and $\vec{b}$ is $\vec{0}$.

26. What is the image of the point (6, 3, –4) with respect to yz-plane? [2008(A)]

Ans. The image of (6, 3, –4) with respect ... (–6, 3, –4).

27. Find the value of k for which the line $\frac{x-2}{3} = \frac{1-y}{k} = \frac{z-1}{4}$ is parallel to the plane $2x + 6y + 3z - 4 = 0$. [2008(A)]

Ans. The line $\frac{x-2}{3} = \frac{1-y}{k} = \frac{z-1}{4}$ is parallel to the plane

$2x + 6y + 3z - 4 = 0$.

$\therefore 2\times 3 + 6\times(-k) + 3\times 4 = 0$

$\Rightarrow 6k = 18 \Rightarrow k = 3$.

28. Write a point on the line $\frac{x-2}{3} = \frac{y+2}{-3} = \frac{z-2}{3}$ [2004(A)]

Ans. A point on the given line is (2, –2, 2).

29. Write the equation of the plane passing through (3, –6, –9) and parallel to xz –plane. [2004(A)]

Ans. The required equation of the plane is $y + 6 = 0$

30. Write the angle between $\vec{a}$ and $\vec{c}$ if $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \frac{1}{2}\vec{c}$ [2004(A)]

Ans. $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \frac{1}{2}\vec{c}$

$\Rightarrow \left(\vec{a}\cdot\vec{c}\right)\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c} = \frac{1}{2}\vec{c}$

$\Rightarrow \left(\vec{a}\cdot\vec{c}\right) = 0$

$\Rightarrow$ Angle between $\vec{a}$ and $\vec{c} = 90°$.

31. Write a vector normal to the plane $2x - 7y = 5z$ [2003(A

Ans. The equation of the plane can be written

$2x - 7y - 5z = 0$

A vector normal to the above plane is

$2\hat{i} - 7\hat{j} - 5\hat{k}$

32. Write the value of $(\hat{i} - \hat{j} + 2\hat{k}).(\hat{i} + \hat{j})$ [20

Ans. $(\hat{i} - \hat{j} + 2\hat{k}).(\hat{i} + \hat{j}) = 1 - 1 + 0 = 0$

33. What is the unit vector in the direct... the vector $3\hat{i} + 4\hat{j}$? [2

Ans. Unit vector in the direction of

$$3\hat{i} + 4\hat{j} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{9+16}} = \frac{3\hat{i} + 4\hat{j}}{5}$$

$x + 3y + 6 + k(3x - y - 4z) = 0$

If this plane passes through (1,1,1) then

$1 + 3 + 6 + k(3 - 1 - 4) = 0$

$\Rightarrow \quad 10 - 2k = 0 \Rightarrow k = 5$

$\therefore$ Equation of the plane is

$x + 3y + 6 + 5(3x - y - 4z) = 0$

$\Rightarrow \quad 16x - 2y - 20z + 6 = 0$

$\Rightarrow \quad 8x - y - 10z + 3 = 0$

GROUP - C

1. If $\vec{a} = 2\hat{i} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$, then find the vector $\vec{r}$ which satisfies $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$. **[*2018(A)***]

Ans. Given $\vec{a} = 2i + k$, $\vec{b} = i + j + k$ and

$\vec{c} = 4i - 3j + 7k$.

Now, $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

$\Rightarrow \quad (\vec{r} - \vec{c}) \times \vec{b} = 0$

$\Rightarrow \quad (\vec{r} - \vec{c}) \parallel \vec{b}$

$\Rightarrow \quad \vec{r} - \vec{c} = \lambda \vec{b}$

$\Rightarrow \quad \vec{r} = \vec{c} + \lambda \vec{b}$

$= (4i - 3j + 7k) + \lambda(i + j + k)$

$= (4 + \lambda)i + (-3 + \lambda)j + (7 + \lambda)k$

But $\vec{r} \cdot \vec{a} = 0$

$\Rightarrow \quad 2(4 + \lambda) + (7 + \lambda) = 0$

$\Rightarrow \quad 3\lambda = -15 \Rightarrow \lambda = -5$

$\therefore \quad \vec{r} = -i - 8j + 2k$

2. Find the shortest distance between the lines $\dfrac{x-3}{3} = \dfrac{y-8}{-1} = \dfrac{z-3}{1}$ and $\dfrac{x+3}{-3} = \dfrac{y+7}{2} = \dfrac{z-6}{4}$. Find also the equations of the line of shortest distance. **[*2018(A)***]

Ans. Given lines are

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \alpha \text{ (say)} \qquad \text{...(1)}$$

and $$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \beta (say) \qquad \text{...(2)}$$

Any point on the line (1) and (2) are P ($3 + 3\alpha$, $8 - \alpha$, $3 + \alpha$) and Q ($-3 - 3\beta$, $-7 + 2\beta$, $6 + 4\beta$) respectively.

D.r.s. of PQ are $< 6 + 3\alpha + 3\beta, 15 - \alpha - 4\beta>$

D.r.s. of the lines are $< 3, -1, 1 >$ and $<-3, 2, 4>$ respectively.

PQ is perpendicular to the given lines.

$\therefore \quad 3(6 + 3\alpha + 3\beta) - (15 - \alpha - 2\beta) + 1(-3 + \alpha - 4\beta) = 0$

and $-3(6 + 3\alpha + 3\beta) + 2(15 - \alpha - 2\beta) + 1(-3 + \alpha - 4\beta)$

$\Rightarrow 18 + 9\alpha + 9\beta - 15 + \alpha + 2\beta - 3 + \alpha - 4\beta = 0$

and $-18 - 9\alpha - 9\beta + 30 - 2\alpha - 4\beta - 12 + 4\alpha - 16\beta = 0$

$\Rightarrow 11a + 7b = 0$ and $-7a - 29b = 0$

$\Rightarrow a = b = 0$

$\therefore$ co-ordinates P and Q are (3, 8, 3) and (−3, − 7, 6) respectively .

The shortest distance

$= PQ = \sqrt{62 + 15^2 + 3^2} = 3\sqrt{30}$

and the equation of the line of shortest distance is

$$\frac{x-3}{-6} = \frac{y-8}{-15} = \frac{z-3}{3} \Rightarrow \frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

3. Show that

$$\left[\vec{a}+\vec{b}\;\vec{b}+\vec{c}\;\vec{c}+\vec{a}\right] = 2\left[\vec{a}\,\vec{b}\,\vec{c}\right]$$ **[2018 (A)]**

Ans. $\left[\vec{a}+\vec{b}\;\vec{b}+\vec{c}\;\vec{c}+\vec{a}\right]$

$= \left(\vec{a}+\vec{b}\right) \cdot \left\{\left(\vec{b}+\vec{c}\right) \times \left(\vec{c}+\vec{a}\right)\right\}$

$= \left(\vec{a}+\vec{b}\right) \cdot \left\{\vec{a} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\right\}$

$= \left(\vec{a}+\vec{b}\right) \cdot \left\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}\right\} \left[\because \vec{c} \times \vec{c} = \vec{0}\right]$

$= \vec{a} \cdot \left(\vec{b} \times \vec{c}\right) + \vec{a} \cdot \left(\vec{b} \times \vec{a}\right) + \vec{a} \cdot \left(\vec{c} \times \vec{a}\right)$

$+ \vec{b} \cdot \left(\vec{b} \times \vec{c}\right) + \vec{b} \cdot \left(\vec{b} \times \vec{a}\right) + \vec{b} \cdot \left(\vec{c} \times \vec{a}\right)$

$= \vec{a} \cdot \left(\vec{b} \times \vec{c}\right) + \vec{b} \cdot \left(\vec{c} \times \vec{a}\right)$

[$\because$ Scalar triple product vanishes if any t of them are equal].

$= \vec{a} \cdot \left(\vec{b} \times \vec{c}\right) + \left(\vec{b} \times \vec{c}\right) \cdot \vec{a}$

[Dot and cross product can be interchan

$= \vec{a} \cdot \left(\vec{b} \times \vec{c}\right) + \vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$

[$\because$ Dot product is commutative]

$= 2\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = 2\left(\vec{a}\,\vec{b}\,\vec{c}\right)$

10. Obtain the volume of the parallelopiped whose sides are vectors $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$, $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$. Also find the vector $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$. **[2013 (A)**

Ans. Given edges of the parallelopipead are

$\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$,

$\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$.

Now $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$

$$=\begin{vmatrix}2 & -3 & 4\\ 1 & 2 & -1\\ 3 & -1 & 2\end{vmatrix}$$

$= 2(4-1)+3(2+3)+4(-1-6)$

$= 6+15-28=-7$

$\therefore$ Volume of the parallelopiped = 7 cubic units.

Again $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\left(\vec{a}\times\vec{c}\right)\vec{c}-\left(\vec{b}\times\vec{c}\right)\vec{a}$

$=\left\{\left(2\hat{i}-3\hat{j}+4\hat{k}\right)\cdot\left(3\hat{i}-\hat{j}+2\hat{k}\right)\right\}$

$\left(\hat{i}-2\hat{j}-\hat{k}\right)-\left\{\left(\hat{i}+2\hat{j}-\hat{k}\right)\cdot\left(3\hat{i}-\hat{j}+2\hat{k}\right)\right\}$

$\left(2\hat{i}-3\hat{j}-4\hat{k}\right)$

$=(6+3+8)\left(\hat{i}-2\hat{j}-\hat{k}\right)-(3-2-2)$

$\left(2\hat{i}-3\hat{j}+4\hat{k}\right)$

$=\left(17\hat{i}+34\hat{j}-17\hat{k}\right)-(-1)\left(2\hat{i}-3\hat{j}+4\hat{k}\right)$

$=19\hat{i}+31\hat{j}-13\hat{k}$

11. Prove that the four points with position vectors $4\hat{i}+5\hat{j}+\hat{k}$, $-\hat{j}-\hat{k}$, $3\hat{i}+9\hat{j}+4\hat{k}$ and $-4\hat{j}+4\hat{j}+4\hat{k}$ are coplanar. **[*2012(A)***

Ans. Let the points are A, B, C, D with position vector

$4\hat{i}+5\hat{j}+\hat{k}$, $-\hat{j}-\hat{k}$, $3\hat{i}+9\hat{j}+4\hat{k}$ and $+4\hat{j}+4\hat{j}+4\hat{k}$ respectively.

Now $\overrightarrow{AB}=-4\hat{i}-6\hat{j}-2\hat{k}$

$\overrightarrow{AC}=-4\hat{i}-6\hat{j}-2\hat{k}$

and $\overrightarrow{AD}=-8\hat{i}-\hat{j}+3\hat{k}$

Now $\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)$

$$=\begin{vmatrix}-4 & -6 & -2\\ -1 & 4 & 3\\ -8 & -1 & 3\end{vmatrix}$$

$=(-4)(12+3)+6(-3+24)-2(1+32)$

$=-60+126-66=0$

Thus A, B, C and D are coplanar.

12. Prove that the lines $x+5y+2z+3=0=3x+2y+z-2$ and $\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2}$ are coplanar. Find their point of intersection and the equation of the plane in which they lie. **[*2011(A)***

Ans. Given equations are

$\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2}$ ... (1)

and $x+5y+2z+3=0=3x+2y+z-2$ ... (2)

We know that two lines are coplanar if they are either parallel or intersecting.

Let $\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2}=r$

Any point on (1) has co-ordinates

$(3r-5, r-4, -2r+7)$

These two lines will intersect if

$3r-5+5(r-4)+2(-2r+7)+3=0$ ... (3)

and $3(3r-5)+2(r-4)+(-2r+7)-2=0$... (4)

are consistent.

From (3) we get $4r-8=0 \Rightarrow r=2$

which satisfies equation (4).

Thus two lines are intersecting and hence coplanar.

The point of intersection is (1, –2, 3).

Let us write the equations (2) in symmetrical form.

Now $x+5y+2z+3=0$

and $3x+2y+z-2=0$

$\Rightarrow\quad 6x+4y+2z-4=0$

$(-)\ x+5y+2z+3=0$

$5x-y-7=0$

$\Rightarrow\quad x=\frac{y+7}{5}$

Again we have

$15x+10y+5z-10=0$

$2x+10y+4z+\ 6=0$

$(-)$

$13x+z-16=0$

$\Rightarrow\quad x=-\frac{(z-16)}{-13}$ is the symmetrical form of the line (2).

Equation of the plane containing these two lines is

$$\begin{vmatrix} x+5 & y+4 & z-7 \\ 3 & 1 & -2 \\ 1 & 5 & -13 \end{vmatrix} = 0$$

$\Rightarrow$ $(x+5)(-3) - (y+4)(-37) + (z-7)(14) = 0$

$\Rightarrow$ $-3x + 37y + 14z - 15 + 148 - 98 = 0$

$\Rightarrow$ $-3x + 37y + 14z + 35 = 0$

13. Prove that

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$$

and hence prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$, $\vec{b}\times\left(\vec{c}\times\vec{a}\right)$ and $\vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are coplanar. *[2011(A)]*

Ans. $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$

$+\left(\vec{b}.\vec{a}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{a}+\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}$

$= 0$ ($\because$ Dot product is commutative)

Let $\vec{p} = \vec{a}\times\left(\vec{b}\times\vec{c}\right)$, $\vec{q} = \vec{b}\times\left(\vec{c}\times\vec{a}\right)$

$\vec{r} = \vec{c}\times\left(\vec{a}\times\vec{b}\right)$

Now $\vec{p}+\vec{q}+\vec{r}=0$

$\vec{p} = -\left(\vec{q}+\vec{r}\right)$

$\Rightarrow \vec{p}\times\vec{q} = -\vec{q}\times\vec{q} - \vec{r}\times\vec{q}$

$= \vec{q}\times\vec{r}$ $\left(\because \vec{q}\times\vec{q} = 0\right)$

$\Rightarrow \left(\vec{p}\times\vec{q}\right).\vec{r} = \left(\vec{q}\times\vec{r}\right).\vec{r} = 0 \Rightarrow \left[\vec{p}\,\vec{q}\,\vec{r}\right] = 0$

$\Rightarrow \vec{p}, \vec{q}$ and $\vec{r}$ are coplanar.

14. If the vertices A, B, C of a triangle ABC are at (1, 1, 2), (2, 2, 3), (3, –1, 1) respectively, then using vector method find the area of the triangle. *[2011(A)]*

Ans. Vertices of Δ ABC are A(1, 1, 2), B(2, 2, 3) and C(3, –1, –1) respectively.

Let O is the origin.

$\therefore$ Position vector of A = $i + j + 2k$

Position vector of B = $2i + 2j + 3k$

Position vector of C = $3i - j - k$

$\therefore$ $\overrightarrow{AB}$ = P.V. of B – P.V. of A = $i + j + k$

$\overrightarrow{AC}$ = P.V. of C – P.V. of A = $2i - 2j - 3k$

Now $\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & -2 & -3 \end{vmatrix}$

$= i(-3+2) - j(-3-2) + k(-2-2)$

$= -i + 5j - 4k$

Area of Δ ABC $= \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$= \frac{1}{2}|-i + 5j - 4k|$

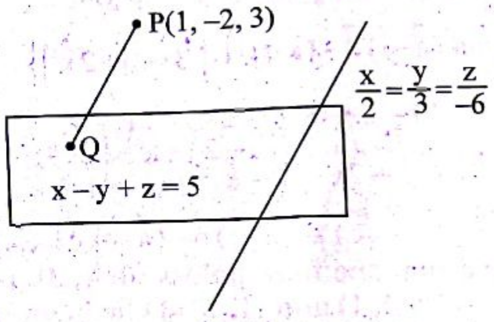
$= \frac{1}{2}\sqrt{1+25+16} = \frac{1}{2}\sqrt{42}$ sq. units.

15. Find the distance of the point (1, –2, 3) from the plane x – y + z = 5 measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$. *[2010(A)]*

Ans. Given point is P(1, –2, 3).

Equation of the plane is $x - y + z = 5$

and line is $\frac{x}{2}=\frac{y}{3}=\frac{z}{6}$.



As PQ is parallel to the given line.

Drs of PQ = <2, 3, –6>

Now equation of PQ is

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}=k \text{ (say)}$$

Any point on the line PQ has coordinates $(2k+1, 3k-2, -6k+3)$.

Let coordinates of Q $= (2k+1, 3k-2, -6k+3)$

Now Q is a point on the plane.

$\therefore$ $2k + 1 - (3k-2) + (-6k+3) = 5$

$\Rightarrow$ $-7k + 1 = 0 \Rightarrow k = \frac{1}{7}$

Hence coordinates of Q $= \left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$

$\therefore$ The required distance = PQ

$$= \sqrt{\left(1-\frac{9}{7}\right)^2+\left(-2+\frac{11}{7}\right)^2+\left(3-\frac{15}{7}\right)^2}$$

$$= \sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$$

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16. **For $\vec{a}=\hat{i}-\hat{j}, \vec{b}=\hat{i}-2\hat{k}, \vec{c}=\hat{j}+\hat{k}$, obtain $\vec{a}\times(\vec{b}\times\vec{c})$ and verify the formula $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$.** *[2009(A)*

Ans. Given vectors are

$$\vec{a}=\hat{i}-\hat{j}, \vec{b}=\hat{i}-2\hat{k}, \vec{c}=\hat{j}+\hat{k}$$

$$\text{Now } \vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}(2) - \hat{j}(1) + \hat{k}(1) = 2\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}(-1) - \hat{j}(1) + \hat{k}(-1+2) = -\hat{i} - \hat{j} + \hat{k}$$

Again $(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$$= (0-1+0)(\hat{i}-2\hat{k}) - (1+0+0)(\hat{j}+\hat{k})$$

$$= -\hat{i} + 2k - \hat{j} - \hat{k} = -\hat{i} - \hat{j} + \hat{k}$$

$\therefore$ $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$ (Verified)

17. **Prove that the four points (0, 4, 3), (–1, –5, – 3), (–2, –2, 1) and (1, 1, –1) lie in one plane. Find the equation of the plane.** *[2009(A)*

Ans. Given points are A(0, 4, 3), B(–1, –5, –3), C(–2, –2, 1) and D(1, 1, –1).

We know that 4 points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ are coplanar

$$\text{iff } \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}=0$$

Applying the condition for given points we have

$$\begin{vmatrix} -1-0 & -5-4 & -3-3 \\ -2-0 & -2-4 & 1-3 \\ 1-0 & 1-4 & -1-3 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -9 & -6 \\ -2 & -6 & -2 \\ 1 & -3 & -4 \end{vmatrix} = 6\begin{vmatrix} -1 & 3 & 3 \\ -2 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 6[(-1)(3) - 3(-5) + 3(-4)] = 0$$

$\therefore$ The given points are coplanar. Equation of the plane containing them is

$$\begin{vmatrix} x-0 & y-4 & z-3 \\ -1 & -9 & -6 \\ -2 & -6 & -2 \end{vmatrix}=0$$

$\Rightarrow$ $x(-18) - (y-4)(-10) + (z-3)(-12) = 0$

$\Rightarrow$ $x(-18) - (y - 4)(-10) + (z - 3)(-12) = 0$ …

$\Rightarrow$ $-18x + 10y - 40 - 12z + 36 = 0$

$\Rightarrow$ $-18x + 10y + 12z + 4 = 0$

$\Rightarrow$ $18x - 10y + 6z + 2 = 0$

$\Rightarrow$ $9x - 5y + 6z + 2 = 0$

18. **A variable plane is at a constant distance 3r from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = r^{-2}$.** *[2008(A)*

Ans. Let the plane meets the x-axis, y-axis and z-axis at A(α, 0, 0), B(0, β, 0) and C(0, 0, γ) respectively.

$\therefore$ Equation of the plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$$

But its distance from origin is 3r.

$$\Rightarrow \quad 3r = \left|\frac{-1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}}\right|$$

$$\Rightarrow \quad \frac{1}{9r^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} \quad \text{... (1)}$$

Now let coordinates of the centroid are (x_1, y_1, z_1).

$$\therefore \quad x_1 = \frac{\alpha+0+0}{3}, y_1=\frac{\beta+0+0}{3}, z_1=\frac{\gamma+0+0}{3}$$

$\Rightarrow$ $\alpha = 3x_1, \beta = 3y_1, \gamma = 3z_1$

$\therefore$ Equation (1) gives

$$\frac{1}{9x_1^2}+\frac{1}{9y_1^2}+\frac{1}{9z_1^2}=\frac{1}{9r^2}$$

$$\Rightarrow \quad \frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}=\frac{1}{r^2}$$

$\therefore$ The locus of the centroid is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=r^2 \quad \text{(Proved)}$$

19. **Show that lines $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ and $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ intersect. Find the coordinates of the point of intersection of the lines and the plane in which the lines lie.** *[2008(A)*

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